Perfect set property

Michal Staš

Department of Mathematics and Theoretical Informatics
Faculty of Electrical Engineering and Informatics, TU in Košice

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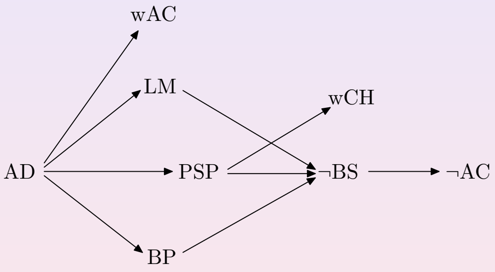
Hejnice

Problem

Does hold true **PSP** in the theory **ZF** + **wAC** + **LM** + **BP**?



wAC: Weak Axiom of Choice
AC: Axiom of Choice
AD: Axiom of Determinacy
BS: there exists a Bernstein set

PSP: every uncount. set of R contains a perfect set
LM: every set of R is Lebesgue measurable
BP: every set of R possesss the Baire property
wCH: there is no set X such that $\aleph_0 < |X| < c$

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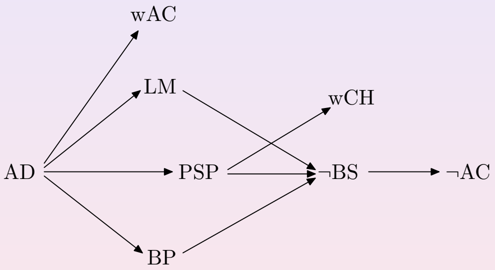

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Useful notions:

- The **Weak Axiom of Choice wAC** says that for any countable family of non-empty subsets of a given set of power $2^{\aleph_0}$ there exists a choice function.
- A subset A is called a **Marczewski null set** or $(S)_0$-set if every perfect set $P \subseteq X$ has a perfect subset Q which misses A. In **ZF** $+$ **wAC** we can verify that the class of all $(S)_0$-sets is a σ-ideal.
- A subset of a perfect Polish space X is called a **totally imperfect** if it contains no perfect subset.

$$[\mathbb{R}]^{\leq\aleph_0} \subseteq \mathcal{S}_0 \subseteq \mathcal{TI}$$

- A set $B \subseteq X$ is called a **Bernstein set** if $|B| = |X \setminus B| = \mathfrak{c}$ and neither B nor $X \setminus B$ contains a perfect subset.

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Theorem 1

If there is no Bernstein set then $\mathcal{S}_0 = \mathcal{TI}$.

- we shall need an auxiliary result

Lemma 2

If there is no Bernstein set on the real line then there is no Bernstein set on the Cantor space ${}^{\omega}2$.

Proof of Lemma 2:

Take the mapping $\varphi : {}^{\omega}2 \to [0,1]$ given by $\varphi(v) = \Sigma_n 2^{-n+1} v(n)$.

- φ is continuous,
- $\varphi(v) \in \mathbb{Q} \cap [0,1]$ if and only if v is an eventually periodic sequence in ${}^{\omega}2$.

If $X \subseteq {}^{\omega}2$ is a Bernstein set, then $\varphi[X] \subseteq [0,1]$ is Bernstein set.

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Let $X \subseteq \mathbb{R}$ be an uncountable totally imperfect set and P being any perfect subset of $\mathbb{R}$.

- if $|X \cap P| \leq \omega$, then there exists a perfect set $Q \subseteq P$ such that $X \cap Q = \emptyset$

In the next, we shall assume that $X \cap P$ is uncountable set.

- fix an enumeration of basic open sets and take maximal open set U such that $X \cap P \cap U$ is countable,
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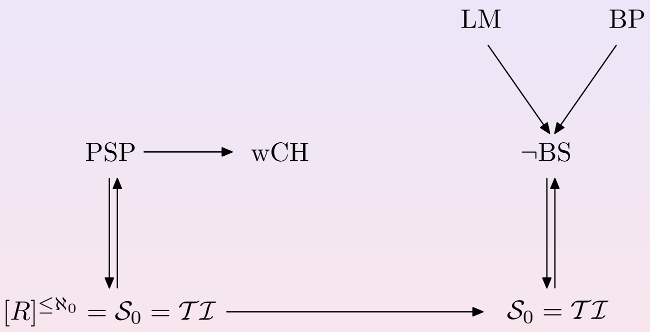
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Diagram 2

Remark in ZF+DC (A. B. Kharazishvili [6])

If there exists a totally imperfect set of reals of cardinality $\mathfrak{c}$, then there exists a Lebesgue non-measurable set of reals.

- this statement one can prove in the theory **ZF+wAC** and Luzin Theorem is essentially exploited for its proof

Theorem 3 in ZF+wAC (N. N. Luzin, see e.g. [5])

Let X, Y be Polish spaces, μ being a Borel measure on X. A function $f : X \to Y$ is μ-measurable if and only if for any positive ε there exists a μ-measurable set $A \subseteq X$ such that $\mu(A) < \varepsilon$ and $f|(X \setminus A)$ is continuous.

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Theorem 4

In the theory **ZF** + **wAC** + **LM** any totally imperfect set of reals has cardinality strictly smaller than $\mathfrak{c}$.

Proof of Theorem 4:

Let $X \subseteq \mathbb{R}$ be a totally imperfect set of cardinality $\mathfrak{c}$ and let $f : \mathbb{R} \to X$ be a bijection.

- Supposing that f is Lebesgue measurable, there exists a measurable set $A \subseteq \mathbb{R}$ with strictly positive measure such that the restriction $f|A$ is continuous.
- The Lebesgue measure is Radon, i.e.

$$\lambda^*(A) = \sup\{\lambda^*(K) : K \subseteq A, K \text{ compact}\}$$

  there exists a compact set K in $\mathbb{R}$ with positive measure.
- K is uncountable and $f|K$ is a homeomorphism.
- $f(K)$ being a subset of X contains a non-empty perfect set, which contradicts the assumption of X.

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In the theory **ZF** + **wAC** + **LM** the following assertions are equivalent:

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Assume that X, Y are metric separable spaces. A function $f : X \longrightarrow Y$ is Baire measurable if and only if there exists a meager set $D \subseteq X$ such that $f|(X \setminus D)$ is continuous. Especially, for any Borel measurable, i.e. for analytically representable function f there exists a meager set $D \subseteq X$ such that $f|(X \setminus D)$ is continuous.

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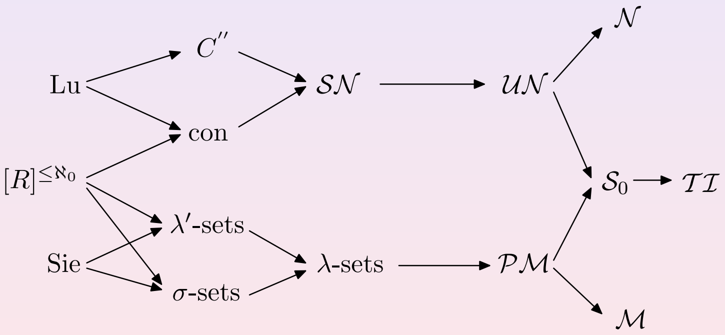

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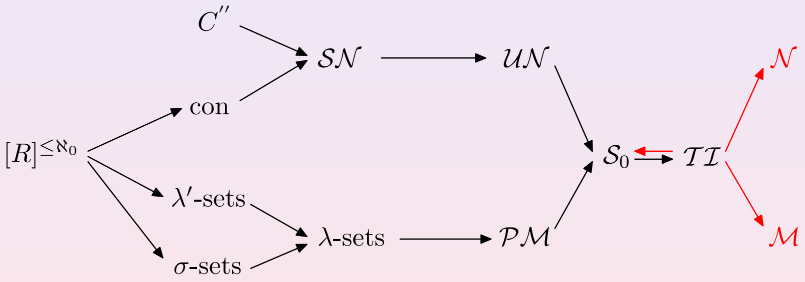

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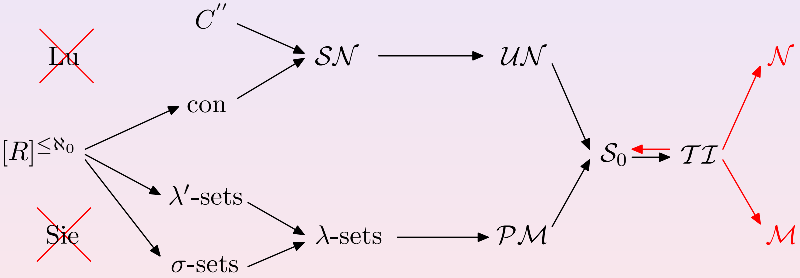

Diagram 5

Borel Conjecture and Generalized Borel Conjecture

Borel Conjecture [1919]

Every set of reals of strong measure zero is countable, i.e. $\mathcal{SN} = [\mathbb{R}]^{<\aleph_1}$.

- the Borel Conjecture is neither provable nor refutable in **ZFC**, mainly by a construction of a model of **ZFC** by R. Laver [7],
- P. Corazza [3] showed that the **Generalized Borel Conjecture**, saying $\mathcal{SN} = [\mathbb{R}]^{<\mathfrak{c}}$, is also independent of **ZFC**,
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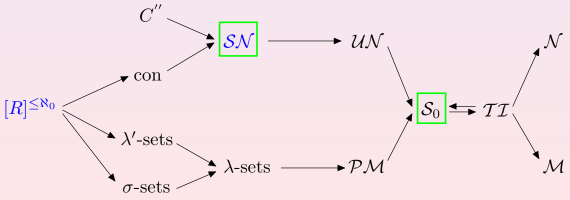

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The Cichoń Diagram - T. Barto., H. Judah and S. Shelah [2]

- described the relationship between the following sentences in the Cichoń Diagram

- $A(m) \equiv$ unions of fewer than $2^{\aleph_0}$ null sets is null,
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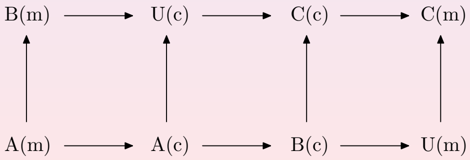

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Theorem 9 in ZF+wAC (F. Galvin, J. Mycielski, R. M. Solovay [4])

A set $A \subseteq \mathbb{R}$ has strong measure zero if and only if for every meager set $F \subseteq \mathbb{R}$ we have $A + F \neq \mathbb{R}$.

Theorem 10

In the theory $\mathbf{ZF} + \mathbf{wAC} + \mathbf{LM} + \mathbf{BP}$ hold true

$$\mathbf{wCH} \to A(m), \quad B(c) \to \mathcal{SN} = [\mathbb{R}]^{<\mathfrak{c}}.$$

Proof of Theorem 10:

- If $\neg A(m)$ then there exists a family $\mathcal{F} \subseteq \mathcal{N}$ of cardinality fewer than $\mathfrak{c}$ such that $\bigcup \mathcal{F} \notin \mathcal{N}$. By **wAC** $\mathcal{N}$ is a σ-ideal, thus the family $\mathcal{F}$ cannot be countable. $\neg A(m) \to \neg\mathbf{wCH}$.
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Thanks for your attention!

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References

- Baire R., *Sur la théorie des fonctions discontinues,* C. R. Acad. Sci. Paris **129** (1899), 1010–1013.
- Bartoszyński, T., Judah, H., Shelah, S., *The Cichoń Diagram,* Journal of Symbolic Logic **58** (1993), 401–423.
- Corazza P., *The generalized Borel conjecture and strongly proper orders,* Trans. Amer. Math. Soc. **316** (1989), 115–140.
- Galvin F., Mycielski J., Solovay R. M., *Strong measure zero sets,* Notices Amer. Math. Soc. **26** (1973), A-280.

References

- Halmos P. R., MEASURE THEORY, Van Nostrand, New York 1950.
- Kharazishvili A. B., *Strange Functions in Real Analysis,* Second edition, Marcel Dekker Inc., New York, 2000.
- Laver R., *On the consistency of Borel's conjecture,* Acta Math. **137** (1976), 151–169.
- Shelah S., *Can you take Solovay inaccessible away?,* Israel J. Math. **48** (1984), 1–47.